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AUTHOR(S):

Misawa, Masashi

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LOCAL REGULARITY FOR THE EVOLUTIONARY P-LAPLACE OPERATOR AND ITS APPLICATION TO THE P-HARMONIC FLOWS

Masashi Misawa^(*)

Department of Mathematics, Faculty of Sciences, Kumamoto University, 2-39-1
Kurokami, Kumamoto-shi, Kumamoto 860-8555, Japan

1 Introduction

In this note we report on a local regularity for the evolution of p -harmonic maps, the p -harmonic flow, in the super- and sub- quadratic cases, which has been recently obtained by the author.

Let \mathcal{N} be a n -dimensional smooth compact Riemannian manifold without boundary and isometrically embedded in \mathbb{R}^l ($l > n$). For a smooth map u from time-space region $\mathbb{R}_\infty^m := (0, \infty) \times \mathbb{R}^m$ ($m \geq 2$) to \mathbb{R}^l we consider the quasilinear parabolic type system of 2nd-ordered partial differential equations

$$(1.1) \quad \begin{cases} \partial_t u - \operatorname{div}(|Du|^{p-2} Du) = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \subset \mathbb{R}^l. \end{cases}$$

In this note we study a local regularity of solutions to the p -harmonic flow (1.1). Here $p > 1$, and $u = (u^i)$, $i = 1, \dots, l$, is a \mathbb{R}^l -valued function, $Du = (D_\alpha u^i)$ is the gradient of a map u with partial derivatives $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, \dots, m$, and $|Du|^2 = \sum_{\alpha=1}^m \sum_{i=1}^l (D_\alpha u^i)^2$, and $A(u)(Du, Du)$ is the second fundamental form of $\mathcal{N} \subset \mathbb{R}^l$ (provided that, if necessary, the manifold \mathcal{N} is assumed to be orientable). The solution of (1.1) is the trajectory of negative direction gradient flow of the p -energy

$$(1.2) \quad E(u) = \int_{\mathbb{R}^m} \frac{1}{p} |Du|^p dx$$

defined for maps u from \mathbb{R}^m to $\mathcal{N} \subset \mathbb{R}^l$. A critical point of the p -energy is prescribed as a solution of the Euler-Lagrange equation

$$(1.3) \quad \begin{cases} -\operatorname{div}(|Du|^{p-2} Du) = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \subset \mathbb{R}^l. \end{cases}$$

and is named the p -harmonic map.

Here our interest is to have the restriction that the image of maps is imposed on the manifold \mathcal{N} , yielding the second fundamental form of \mathcal{N} in the corresponding equations. Now we explicitly look at the second fundamental form of \mathcal{N} in \mathbb{R}^l .

First we simply derive the Euler-Lagrange equation of (1.2) and the gradient flow (1.1). Let u be a smooth map from \mathbb{R}^m to \mathcal{N} and ϕ a smooth \mathbb{R}^l -vector valued function on \mathbb{R}^m with compact support. Let $\Pi : \mathbb{R}^l \supset \mathcal{O}(\mathcal{N}) \rightarrow \mathcal{N} \subset \mathbb{R}^l$ be the nearest point projection from a tubular neighborhood $\mathcal{O}(\mathcal{N}) \subset \mathbb{R}^l$ of \mathcal{N} , to \mathcal{N} . For any sufficient small number τ , $|\tau| \ll |\phi|_\infty$, the map $u + \tau\phi$ has its value in $\mathcal{O}(\mathcal{N})$ and so, $\Pi(u + \tau\phi) \in \mathcal{N}$ is a admissible

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comparison map. The first variation (Gâteaux derivative) is computed by integration by parts as

(1.4)

$$\left. \frac{d}{d\tau} E(\Pi(u + \tau\phi)) \right|_{\tau=0} = \int_{\mathbb{R}^m} \left(-\operatorname{div}(|Du|^{p-2} Du) + |Du|^{p-2} \frac{d^2\Pi}{du^2}(u)(Du, Du) \right) \cdot \phi \, dx.$$

Thus, the Euler-Lagrange equation (1.3) is the first variational formula, (1.4)=0. For smooth maps $u \in C^\infty(\mathbb{R}^m, \mathcal{N})^\dagger$, its gradient-like vector field $\nabla E(u)$ of the p -energy is formally defined as

$$\langle \nabla E(u), \phi \rangle^\ddagger = \left. \frac{d}{d\tau} E(\Pi(u + \tau\phi)) \right|_{\tau=0}$$

and thus, by (1.4)

$$\nabla E(u) = -\operatorname{div}(|Du|^{p-2} Du) + |Du|^{p-2} \frac{d^2\Pi}{du^2}(u)(Du, Du)$$

and so, the solution-curve $\{u(t)\} \subset C^\infty(\mathbb{R}^m, \mathcal{N})$, $0 \leq t < \infty$, of its negative direction gradient vector field is the solution to the differential equation (1.1).

Next let $\mathbb{R}^l = \mathcal{T}_u\mathcal{N} \oplus (\mathcal{T}_u\mathcal{N})^\perp$ be the orthogonal decomposition of \mathbb{R}^l with respect to the tangent space $\mathcal{T}_u\mathcal{N}$ at each $u \in \mathcal{N}$. The corresponding orthonormal basis is $(e_1(u), \dots, e_n(u))$ of the tangent space $\mathcal{T}_u\mathcal{N}$ and $(e_{n+1}(u), \dots, e_l(u))$ of its orthogonal complement $(\mathcal{T}_u\mathcal{N})^\perp$. Then the second fundamental form can be written as

$$A(u)(Du, Du) = \sum_{j=n+1}^l \sum_{i=1}^l \left(Du \cdot Du^i \frac{\partial e_j}{\partial u^i}(u) \right) e_j(u)$$

and thus, $A(u)(Du, Du) \in (\mathcal{T}_u\mathcal{N})^\perp$ for each $u \in \mathcal{N}$. On the other hand, $\partial_t u \in \mathcal{T}_u\mathcal{N}$ and $D_\alpha u \in \mathcal{T}_u\mathcal{N}$, $\alpha = 1, \dots, m$, because the image of maps $u = u(t, x)$ restricted on the manifold \mathcal{N} . Thus, making the Euclidean inner product in \mathbb{R}^l with the equation (1.1) gives

$$|\partial_t u|^2 - \Delta_p u \cdot \partial_t u = 0, \quad \partial_t u \cdot D_\alpha u - \Delta_p u \cdot D_\alpha u = 0,$$

and the crucial formulas for local energy estimates, respectively,

$$\begin{aligned} |\partial_t u|^2 - \operatorname{div}(|Du|^{p-2} Du \cdot \partial_t u) + \partial_t \frac{1}{p} |Du|^p &= 0, \\ \partial_t u \cdot D_\alpha u - \operatorname{div}(|Du|^{p-2} Du \cdot D_\alpha u) + D_\alpha \frac{1}{p} |Du|^p &= 0, \quad \alpha = 1, \dots, m. \end{aligned}$$

In particular, the first formula is integrated on space and yields, through integration by parts,

$$\frac{d}{dt} E(u(t)) = -\|\partial_t u(t)\|_2^2$$

and thus, the p -energy $E(u(t))$ is decreasing along the solution $u(t)$ of the p -harmonic flow. A global in time solution to (1.1) for any initial data may converge to the critical

[†] $C^\infty(\Omega, \mathcal{N})$ is a Banach manifold

[‡] $\langle \nabla E(u), \cdot \rangle$ is a bounded linear functional on a tangent space $\bigcup_{u \in \mathcal{X}} C^\infty(\Omega, \mathcal{T}_u(\mathcal{N}))$ of a Banach manifold $\mathcal{X} := C^\infty(\Omega, \mathcal{N})$.

points of the p -energy, the p -harmonic maps, as time tending to ∞ . This *heat flow method* is originally realized by J. Eells and J. H. Sampson in the harmonic flow case $p = 2$ ([7]). Their fundamental result also holds similarly for the p -harmonic flow under the condition on target that the sectional curvature of \mathcal{N} is non-positive (see [15, 8]).

Without any curvature restriction on the target manifold, there is a blowing up solution at a finite time (see [2] in the case $p = m = 3$). Thus, a weak solution is naturally considered. A weak solution which is locally continuous on time-space together with its gradient is called a regular solution.

Theorem 1 [12] *Let $p = m \geq 2$ and let the initial data be in the set of Sobolev maps $W^{1,p}(\mathcal{M}, \mathcal{N})$ between two smooth, compact Riemannian manifolds \mathcal{M} and \mathcal{N} without boundaries. Then, there exists a global in time weak solution of Cauchy problem for the m -harmonic flow. The solution is regular, except for at most finitely many time slices.*

In the two-dimensional harmonic flow case $p = m = 2$, the solution is smooth except for at most finitely many points [20]. In the case $p = m$, a nice Sobolev type inequality on time-space, referred as Ladyzhenskaya or Nash inequality in $p = m = 2$, plays an important role in regularity estimate.

The global in time existence of partial regular weak solution to the harmonic flow in the case $p = 2$ has been established by M. Struwe et al. in [21, 4]. The crucial ingredient for the result is the so-called small energy regularity estimate as follows : Let $T > 0$ and $X \in \mathbb{R}^m$, and let the *backward* in time heat kernel with pole at (T, X) be

$$G(t, x) = \frac{1}{(4\pi(T-t))^{m/2}} \exp\left(-\frac{|x-X|^2}{4(T-t)}\right), \quad t < T.$$

The scaled energy is defined as

$$I(T, X; r) = r^2 \int_{\{t=T-r^2\}} \frac{1}{2} |Du(t, x)|^2 G(t, x) dx, \quad 0 < r \leq T^{1/2}.$$

The following monotonicity estimate holds true (see [21, Lemma 3.2, pp. 489-490]).

Lemma 2 (monotonicity formula) *Let $p = 2$ and let u be a smooth solution of the harmonic flow (1.1) on $\mathbb{R}_T^m = (0, T) \times \mathbb{R}^m$ for $T > 0$. For any positive $r < \rho \leq T^{1/2}$ it holds that*

$$I(T, X; r) \leq I(T, X; \rho).$$

From the monotonicity estimate of scaled energy and the gradient L^∞ -estimate on *small region* for harmonic flow, the following regularity estimate is obtained (see [21, Proposition 4.1, p. 490 ; Theorem 5.1, its proof, pp. 491-493 ; Theorem 5.3, p. 494] and also [22, Proof of Theorem, pp. 171-172]).

Theorem 3 (small energy regularity) *Let $p = 2$ and let u be a smooth solution of the harmonic flow (1.1) on $\mathbb{R}_T^m = (0, T) \times \mathbb{R}^m$. Then there exist positive constants ϵ_0 and C depending only on m and \mathcal{N} such that the following holds true : If $I(T, X; R) \leq \epsilon_0$ for some $X \in \mathbb{R}^m$ and some positive $R \leq T^{1/2}$, then it holds that*

$$\sup_{(T-(R/4)^2, T) \times B(R/4, X)} |Du| \leq C R^{-1}.$$

There also exist blowing up solutions at a finite time (see [1, 3, 5, 10]).

Based on the a-priori estimate for smooth solutions in Theorem 3 with an appropriate approximation method, it is shown in [4] that, for the Cauchy problem for harmonic flow in the case $p = 2$, there exists a global in time weak solution which is *partial regular* in the sense of regularity outside exceptional closed set. The local regularity estimate has recently been established for the p -harmonic flow in the superquadratic case $p > 2$ (see [16, 17]), which corresponds to the small energy regularity result as in Theorem 3 for the p -harmonic flow.

Now we will present our main result, the *small energy regularity* estimate for the p -harmonic flow.

Theorem 4 *Let λ_0 , B_0 and a_0 be positive numbers satisfying the conditions : In the superquadratic case $p > 2$,*

$$(1.5) \quad \frac{4(p-1)}{p} < \lambda_0 = B_0 < p \quad ; \quad \frac{\lambda_0 - 2}{p-2} < a_0 \leq 1$$

and, in the subquadratic case $\frac{2m}{m+2} < p < 2$,

$$(1.6) \quad p < \lambda_0 = B_0 < \min \left\{ \frac{4}{4-p}, 3 - \frac{2}{p} \right\} \quad ; \quad \frac{2 - \lambda_0}{2-p} < a_0 \leq 1.$$

Let u be a regular solution of (1.1) on \mathbb{R}_T^m for a positive $T < \infty$, satisfying the energy bound

$$p \|\partial_t u\|_{L^2(\mathbb{R}_T^m)}^2 + \sup_{0 < t < T} \|Du(t)\|_{L^p(\mathbb{R}^m)}^p \leq C$$

for a positive number C depending only on m , p and N . Then, there exists a small positive number $R_0 < 1$, depending only on m , p , B_0 and a_0 , and the following holds true : If, for some small positive $R < \min\{R_0, T^{1/\lambda_0}\}$ and some $X \in \mathbb{R}^m$,

$$(1.7) \quad \limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{\{t=T-R^{\lambda_0}\} \times B(r, X)} |Du(t, x)|^p dx \leq 1, \quad \gamma_0 = \frac{p(B_0 - 2)}{p-2},$$

then, the inequality holds

$$(1.8) \quad \sup_{(T-(R/4)^{\lambda_0}, T) \times B(R/4, X)} |Du| \leq C R^{-a_0},$$

where the positive constant C depends only on λ_0 , B_0 , a_0 , m , p and N .

The condition (1.7) is the local regularity criterion for regular solutions with energy boundedness of the p -harmonic flow to be the uniformly locally bounded of gradients as in (1.8) and thus, uniformly locally continuously differentiable (see [6, 13, 14]). The scale order in condition in (1.7) is *almost optimal*, comparing with the corresponding uniform regularity criterion for regular solutions of stationary p -harmonic maps because the exponent γ_0 can be chosen as close to p as possible, by the condition of B_0 in (1.5) or (1.6).

The main ingredients of Theorem 4 are the monotonicity estimates of scaled energy and the L^∞ -estimate of gradients (see [6, 13, 14]), well-combined under a time-space scaling. The technical novelty here is a new *monotonicity* type estimate of a *localized* scaled p -energy, which may be of its own interest. Let us define our localized scaled

p -energy in the following way: Let $T \geq 0$ and $X \in \mathbb{R}^m$ be given, and (t_0, x_0) in the parabolic like envelope

$$\{(t, x) \in (0, \infty) \times \mathbb{R}^m : t - T \geq |x - X|^{\lambda_0}\} \quad ; \quad \lambda_0 > 2.$$

The localized scaled energy is defined as

$$(1.9) \quad E_{\pm}(r) = \frac{1}{\Lambda^p} \int_{\{t=t_0 \pm \Lambda^{2-p}r^2\} \times \mathbb{R}^m} \frac{1}{p} |Du(t, x)|^p \mathcal{B}_{\pm}(t_0, x_0; t, x) \mathcal{C}^q(t, x) dx,$$

where $\Lambda = \Lambda(r)$ is a function of a scale radius r , defined as

$$(1.10) \quad \Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}} \quad ; \quad \lambda_0 = B_0 \text{ is as in (1.5) or (1.6).}$$

The *forward* or *backward* in time Barenblatt like function denoted by \mathcal{B}_+ and \mathcal{B}_- , respectively, are defined as

$$(1.11) \quad \mathcal{B}_{\pm}(t_0, x_0; t, x) = \frac{1}{(\mp t_0 \pm t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{(\mp t_0 \pm t)^{\frac{1}{B_0}}} \right)^a \right)_+^b, \quad \mp t < \mp t_0 \quad ;$$

$a, b > 1$; determined later.

The localized function \mathcal{C} is defined and used as

$$(1.12) \quad \mathcal{C}(t, x) := \left((t - T)^{1/\lambda_0} - |x - X| \right)_+ \quad ; \quad q > 2.$$

We call $E_+(r)$ and $E_-(r)$ the forward and backward localized scaled p -energy, respectively.

The followings are our monotonicity type estimate of scaled energy.

Lemma 5 (backward monotonicity estimate) *Suppose that $t_0 - T \leq 2$. For any regular solution to the p -harmonic flow the following estimate holds for all positive numbers r, ρ , $r^{B_0} = \Lambda(r)^{2-p}r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq \min\{1, (t_0 - T)/2\}$*

$$(1.13) \quad E_-(r) \leq E_-(\rho) + C(\rho^{\mu} - r^{\mu}) + C \int_{t_0 - \rho^{B_0}}^{t_0 - r^{B_0}} \|\mathcal{C}^{\tilde{q}}(t) |Du(t)|^{\hat{p}}\|_{L^{\infty}(B((t_0 - t)^{1/B_0}, x_0))} dt,$$

where $\hat{p} = \max\{2(p-1), 2\}$ and, $\tilde{q} = \min\{q-2, q(p-1)/p\}$, B_0 as in (1.10), and the positive exponent μ depends only on \mathcal{N} , m , p and B_0 , and the positive constant C depends only on the same ones as μ and q .

Lemma 6 (forward monotonicity estimate) *Suppose that $t_0 - T \leq 1$. For any regular solution to the p -harmonic flow the following estimate holds for all positive numbers r, ρ , $r^{B_0} = \Lambda(r)^{2-p}r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq 1$*

$$(1.14) \quad E_+(\rho) \leq E_+(r) + C(\rho^{\mu} - r^{\mu}) + C \int_{t_0 + r^{B_0}}^{t_0 + \rho^{B_0}} \|\mathcal{C}^{\tilde{q}}(t) |Du(t)|^{\hat{p}}\|_{L^{\infty}(B((t - t_0)^{1/B_0}, x_0))} dt,$$

where $\hat{p} = \max\{2(p-1), 2\}$ and, $\tilde{q} = \min\{q-2, q(p-1)/p\}$, B_0 as in (1.10), and the positive constants μ and C have the same dependence as those in Lemma 5.

From Theorem 4, a compactness for regular p -harmonic flows with uniform boundedness of p -energy is obtained (see [21, Theorem 6.1 ; its proof, pp. 494-497] for the harmonic flow). The compactness result will be the key ingredient for the global in time existence of p -harmonic flow, which will be studied in the near future work (refer to [4] for the harmonic flow case).

Theorem 7 (compactness of regular p -harmonic flows) *Suppose that a family $\{u_k\}$ of regular p -harmonic flows on $\mathbb{R}_\infty^m = (0, \infty) \times \mathbb{R}^m$ satisfies the p -energy boundedness with uniform positive constant C*

$$(1.15) \quad p \|\partial_t u_k\|_{L^2(\mathbb{R}_\infty^m)}^2 + \sup_{0 < t < \infty} \|Du_k(t)\|_{L^p(\mathbb{R}^m)}^p \leq C$$

and converges to a limit map u in the sense

$$(1.16) \quad u_k \longrightarrow u \quad \text{weakly } * \text{ in } L^\infty(0, T; W^{1,p}(\mathbb{R}_\infty^m, \mathbb{R}^l)),$$

$$(1.17) \quad Du_k \longrightarrow Du \quad \text{weakly in } L^p(\mathbb{R}_\infty^m, \mathbb{R}^{ml}),$$

$$(1.18) \quad \partial_t u_k \longrightarrow \partial_t u \quad \text{weakly in } L^2(\mathbb{R}_\infty^m, \mathbb{R}^l).$$

Then, the limit map u is a global weak solution on \mathbb{R}_∞^m of the p -harmonic map heat flow such that $u \in \mathcal{N}$ almost everywhere in \mathbb{R}_∞^m , and the p -energy boundedness is valid, replacing u_k by u in (1.15). Moreover, the limit map u is partial regular in the sense : Let $R_0 < 1$ be a positive number, defined in Theorem 4 and a subset $\mathcal{S} \subset \mathbb{R}_\infty^m$ be defined as

$$(1.19) \quad \mathcal{S} := \left\{ (\tau, x_0) \in \mathbb{R}_\infty^m : \text{for all positive } R < \min\{R_0, \tau^{1/\lambda_0}\}, \right. \\ \left. \limsup_{k \rightarrow \infty} \left(\limsup_{r \searrow 0} \tau^{\gamma_0 - m} \int_{\{t=\tau-R^{\lambda_0}\} \times B(r, x_0)} |Du_k(t, x)|^p dx \right) \geq 1 \right\}.$$

Then, \mathcal{S} is closed in \mathbb{R}_∞^m and u and its gradient Du are locally in time-space continuous in the complement $\mathbb{R}_\infty^m \setminus \mathcal{S}$. The size of \mathcal{S} is also estimated by the Hausdorff measure : Let γ_0, λ_0, B_0 and a_0 be the same positive numbers as in (1.5) and (1.7) in Theorem 4. The set \mathcal{S} is of at most locally zero m -dimensional Hausdorff measure with respect to the time-space metric $|t|^{1/\gamma_0} + |x|$, $\mathcal{H}^m(\mathcal{S} \cap K) = 0$ for any open subset K compactly contained in \mathbb{R}_∞^m , and, furthermore, for any positive time $\tau < \infty$, the $(m - \gamma_0)$ -dimensional Hausdorff measure of $\{\tau\} \times \mathcal{S}$ with respect to the usual Euclidean metric is locally zero, $\mathcal{H}^{m-\gamma_0}(\{\tau\} \times \mathcal{S} \cap K) = 0$ for any open subset K compactly contained in \mathbb{R}^m .

2 Monotonicity estimate

We demonstrate the monotonicity estimates in the superquadratic case $p > 2$. For brevity, we reset as $\mathcal{C} \equiv 1$.

Let $z_0 = (t_0, x_0) \in (0, T] \times \mathbb{R}^m$. As before, we put

$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad B_0 > \frac{4(p-1)}{p}$$

and let r any positive number in the range $0 < r \leq \min\{1, (t_0)^{1/B_0}\}$. We make a scaling transformation intrinsic to the evolutionary p -Laplace operator (refer to [6, 13, 14])

$$(2.1) \quad t = t_0 + \Lambda^{2-p} r^2 s; \quad x = x_0 + r y; \quad v(s, y) = \frac{u(t_0 + \Lambda^{2-p} r^2 s, x_0 + r y)}{\Lambda r}$$

and, under the scaling transformation

$$t = t_0 - \Lambda^{2-p} r^2 \iff s = -1.$$

Then the scaled solution v is a solution of the scaled equation on $\{s = -1\} \times \mathbb{R}^m$

$$(2.2) \quad \partial_s v - \operatorname{div} \left(|Dv|^{p-2} Dv \right) = -\Lambda r |Dv|^{p-2} A(r\Lambda v)(Dv, Dv)$$

and the scaled p -energy is rewritten as

$$(2.3) \quad E(r) = \int_{\{s=-1\} \times \mathbb{R}^m} \frac{1}{p} |Dv(s, y)|^p \mathcal{B}(s, y) dy \quad ; \quad \mathcal{B}(s, y) = (1 - |y|^a)_+^b$$

by simply computing as

$$\begin{aligned} Dv(s, y) &= \frac{1}{\Lambda} D_x u(t, x) \quad ; \quad \Lambda = r^{\frac{B_0-2}{2-p}} \iff \Lambda^{\frac{p-2}{B_0}} r^{\frac{B_0-2}{B_0}} = 1 \quad ; \\ \mathcal{B}(s, y) dy &= \mathcal{B}(t_0, x_0; t, x) dx. \end{aligned}$$

Our main task in monotonicity estimate is to derive appropriate values of parameters

$$(2.4) \quad B_0 > \frac{4(p-1)}{p} \quad ;$$

$$(2.5) \quad 0 < \delta \leq \frac{B_0(p-2)}{B_0-2} \left(-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} \right).$$

Step 1: differentiation of $E(r)$ on r . Now we compute differentiation of $E(r)$ on r . By the equation (2.2) and integration by parts,

$$\begin{aligned} \frac{d}{dr} E(r) &= \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{p-2} Dv \cdot \frac{d}{dr} Dv \mathcal{B}(s, y) dy \\ &= \int_{\{s=-1\} \times \mathbb{R}^m} \frac{dv}{dr} \cdot \left(-\Delta_p v \mathcal{B}(s, y) - |Dv|^{p-2} Dv \cdot D\mathcal{B}(s, y) \right) dy \\ &= r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} (-s) \left((2-p) r \Lambda^{-1} \Lambda' + 2 \right) |\partial_s v|^2 \mathcal{B}(s, y) dy \\ &\quad - r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} (y \cdot Dv) \cdot \partial_s v \mathcal{B}(s, y) dy \\ &\quad + r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} \left(1 + r \Lambda^{-1} \Lambda' \right) v \cdot \partial_s v \mathcal{B}(s, y) dy \\ (2.6) \quad &+ \int_{\{s=-1\} \times \mathbb{R}^m} \Lambda \left(1 + r \Lambda^{-1} \Lambda' \right) |Dv|^{p-2} v \cdot A(r\Lambda v)(Dv, Dv) \mathcal{B}(s, y) dy \\ &+ ab r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} \left\{ |Dv|^{p-2} |y \cdot Dv|^2 \right. \\ &\quad \left. + \left((2-p) r \Lambda^{-1} \Lambda' + 2 \right) |Dv|^{p-2} (y \cdot Dv) \cdot (s \partial_s v) \right. \\ &\quad \left. - \left(1 + r \Lambda^{-1} \Lambda' \right) |Dv|^{p-2} (y \cdot Dv) \cdot v \right\} \times \\ &\quad \times |y|^{a-2} (1 - |y|^a)_+^{b-1} dy, \end{aligned}$$

where, noting that $\Lambda = r^{(B_0-2)/(2-p)}$, the generator of dilation is computed as

$$\frac{dv}{dr} = r^{-1} \left(- \left(1 + r \Lambda^{-1} \Lambda' \right) v + \left((2-p) r \Lambda^{-1} \Lambda' + 2 \right) s \partial_s v + y \cdot Dv \right)$$

Now each term in (2.6) is separately estimated.

1st term of (2.6). In the first term of (2.6) the coefficient is positive, because, by definition of Λ and $s = -1$

$$(-s) \left((2-p) r \Lambda^{-1} \Lambda' + 2 \right) = B_0 > 0 \iff \Lambda = r^{(B_0-2)/(2-p)}.$$

2nd term of (2.6). By Cauchy's inequality with small $c > 0$ the second term of (2.6) is estimated below by

$$-\frac{c}{2} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |\partial_s v|^2 \mathcal{B} dy - \frac{1}{2c} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |y|^2 |Dv|^2 \mathcal{B} dy.$$

The time-derivative term is absorbed into the first term and, by the support of the Barenblatt like weight \mathcal{B} , $\{y \in \mathbb{R}^m \mid |y| \leq 1\}$, and by Young's inequality with $\delta > 0$, the spatial gradient term is estimated below by

$$(2.7) \quad -C r^{-1} \Lambda^\delta \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{2(p-1)} \mathcal{B} dy - C r^{-1} \Lambda^{-\frac{\delta}{p-2}} \int_{\{s=-1\} \times \mathbb{R}^m} \mathcal{B} dy.$$

3rd term of (2.6). For estimation of the third term of (2.6) we use the Poincaré type inequality with weight of Barenblatt like function § [18, Theorem 5.3.4, p. 134].

Lemma 8

$$(2.8) \quad \int_{\{s=-1\} \times \mathbb{R}^m} |v|^2 \mathcal{B} dy \leq C \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^2 \mathcal{B} dy.$$

By Cauchy's inequality with small $c > 0$ the third term is estimated below by

$$(2.9) \quad -\frac{c}{2} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |\partial_s v|^2 \mathcal{B} dy - \frac{1}{2c} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |v|^2 \mathcal{B} dy,$$

where, by definition of Λ , $1 + r \Lambda^{-1} \Lambda' = (p - B_0)/(p - 2)$. The first time-derivative term is absorbed into that of (2.6). By the support of \mathcal{B} , again, the Poincaré inequality, Lemma 8, and Young's inequality, the second term is bounded below by (2.7).

4th term of (2.6). By the definition of Λ , the fourth term of (2.6) is estimated below as

$$(2.10) \quad -\frac{|p - B_0|}{p - 2} C'(\mathcal{N}) \Lambda \int_{\{s=-1\} \times \mathbb{R}^m} |v| |Dv|^p \mathcal{B} dy \geq -C r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^p \mathcal{B} dy,$$

where the second fundamental form A is bounded by the compactness of target \mathcal{N} and, by scaling back and the compactness of target \mathcal{N} ,

$$\Lambda |v| = \Lambda \frac{1}{\Lambda r} |u| \leq \frac{1}{r} \|u\|_{L^\infty(\mathbb{R}^m)} \leq r^{-1} C(\mathcal{N}).$$

§ Our estimations here remained unchanged, even if v is replaced by $v - \bar{v}$ with weighted integral mean $\bar{v} = \int_{\{s=-1\} \times \mathbb{R}^m} v \mathcal{B} dy / \int_{\{s=-1\} \times \mathbb{R}^m} \mathcal{B} dy$.

By Young's inequality, (2.10) is bounded below for $\delta > 0$ by

$$(2.11) \quad -C r^{-1} \Lambda^\delta \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{2(p-1)} \mathcal{B} dy - C r^{-1} \Lambda^{-\frac{\delta p}{p-2}} \int_{\{s=-1\} \times \mathbb{R}^m} \mathcal{B} dy.$$

5th term of (2.6). The fifth term of (2.6) is clearly nonnegative.

6th term of (2.6). The sixth term of (2.6) appears from the nonhomogeneity of evolutionary p -Laplace operator and is estimated below by Cauchy's inequality as

$$(2.12) \quad -\frac{C}{2} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |\partial_s v|^2 \mathcal{B} dy - \frac{C}{2c} r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{2(p-1)} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2} dy,$$

where, by definition of Λ , $(2-p)r\Lambda^{-1}\Lambda' + 2 = B_0$, as before. The first term of (2.12) is absorbed into that of (2.6). The second term of (2.12) is estimated below by

$$(2.13) \quad -\frac{C}{2\delta} r^{-1} \frac{1}{\Lambda^{2(p-1)}} \|Du(\tau)\|_{L^\infty(\text{supp}\mathcal{B}(\tau))}^{2(p-1)} \Big|_{\tau=t_0-\Lambda^{2-p}r^2},$$

where, by a scaling back,

$$\int_{\mathbb{R}^m} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2} dy < \infty, \\ 1 + \frac{m-2}{a} > -1 \iff a > 0 \quad ; \quad b-2 > -1 \iff b > 1.$$

7th term of (2.6). As in (2.12), the seventh term of (2.6) is bounded below by

$$-C r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |v|^2 \mathcal{B} dy - C r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{2(p-1)} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2} dy,$$

where the first one is the same as the second term in (2.9) and bounded below for $\delta > 0$ by (2.7) and, the second one is the same as in (2.12), together with the first ones of (2.7) and (2.11), estimated below by

$$(2.14) \quad -C r^{-1} (\Lambda^\delta + 1) \frac{1}{\Lambda^{2(p-1)}} \|Du(\tau)\|_{L^\infty(\text{supp}\mathcal{B}(\tau))}^{2(p-1)} \Big|_{\tau=t_0-\Lambda^{2-p}r^2}.$$

Resulting estimation of (2.6). Combining all of the estimations above we have

$$(2.15) \quad \frac{d}{dr} E(r) \geq I - C r^{-1} \left(\Lambda^{-\frac{\delta p}{p-2}} + \Lambda^{-\frac{\delta}{p-2}} \right) - C r^{-1} \frac{1}{\Lambda^{2(p-1)}} (\Lambda^\delta + 1) \|Du(\tau)\|_{L^\infty(\text{supp}\mathcal{B}(\tau))}^{2(p-1)} \Big|_{\tau=t_0-\Lambda^{2-p}r^2},$$

where $\Lambda = r^{(B_0-2)/(2-p)}$, and we put

$$I = \frac{1}{2} B_0 r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} (-s) |\partial_s v|^2 \mathcal{B}(s, y) dy + ab r^{-1} \int_{\{s=-1\} \times \mathbb{R}^m} |Dv|^{p-2} |y \cdot Dv|^2 |y|^{a-2} (1 - |y|^a)_+^{b-1} dy.$$

The terms I is clearly nonnegative. From (2.15) integrated on (r, ρ)

$$\begin{aligned}
 & E(\rho) - E(r) \\
 & \geq -C \int_r^\rho r^{-1} \left(\Lambda^{-\frac{\delta p}{p-2}} + \Lambda^{-\frac{\delta}{p-2}} \right) dr \\
 (2.16) \quad & -C \int_r^\rho r^{-1} \frac{1}{\Lambda^{2(p-1)}} (\Lambda^\delta + 1) \|Du(\tau)\|_{L^\infty(\text{supp } \mathcal{B}(\tau))}^{2(p-1)} \Big|_{\tau=t_0-\Lambda^{2-p} r^2} dr.
 \end{aligned}$$

Step 2 : a uniform bound. We will make a bound of each term in the right hand side of (2.16).

2nd line of (2.16). The first term in the second line of (2.16) is computed as

$$\begin{aligned}
 \int_r^\rho r^{-1} \Lambda^{-\frac{\delta p}{p-2}} dr &= \int_r^\rho r^{-1-\frac{p\delta(B_0'-2)}{(p-2)^2}} dr \\
 &= \frac{(p-2)^2}{p\delta(B_0-2)} \left(\rho^{\frac{p\delta(B_0-2)}{(p-2)^2}} - r^{\frac{p\delta(B_0-2)}{(p-2)^2}} \right),
 \end{aligned}$$

where

$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad \frac{p\delta(B_0-2)}{(p-2)^2} > 0 \iff \delta > 0 \quad ; \quad B_0 > 2.$$

Similarly as above, another term in the second line of (2.16) is

$$\int_r^\rho r^{-1} \Lambda^{-\frac{\delta}{p-2}} dr = \frac{(p-2)^2}{\delta(B_0-2)} \left(\rho^{\frac{\delta(B_0-2)}{(p-2)^2}} - r^{\frac{\delta(B_0-2)}{(p-2)^2}} \right).$$

3rd line of (2.16). The term in the third line of (2.16) is bounded by

$$\begin{aligned}
 & \Lambda = r^{(B_0-2)/(2-p)} \quad ; \\
 & \int_r^\rho r^{-1} \left(-B_0 \Lambda^{2-p} r \right)^{-1} \frac{1}{\Lambda^{2(p-1)}} \times \Lambda^\delta \times \|Du(\tau)\|_{L^\infty(\text{supp } \mathcal{B}(\tau))}^{2(p-1)} \left(-B_0 \Lambda^{2-p} r \right) dr \\
 (2.17) \quad &= \frac{1}{B_0} \int_{t_0-(\Lambda(\rho))^{2-p} \rho^2}^{t_0-(\Lambda(r))^{2-p} r^2} (t_0 - \tau)^{-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)}} \|Du(\tau)\|_{L^\infty(\text{supp } \mathcal{B}(\tau))}^{2(p-1)} d\tau,
 \end{aligned}$$

where by definition of Λ

$$\Lambda = r^{(B_0-2)/(2-p)} \iff (\Lambda(r))^{2-p} r^2 = r^{B_0}$$

and, in the last term a changing of variable is performed

$$\begin{aligned}
 \tau = t_0 - \Lambda^{2-p} r^2 &\iff t_0 - \tau = \Lambda^{2-p} r^2 = r^{B_0} \quad ; \\
 \frac{d\tau}{dr} &= -B_0 \Lambda^{2-p} r \iff d\tau = -B_0 \Lambda^{2-p} r dr.
 \end{aligned}$$

Here the exponent of power of $(t_0 - \tau)$ in (2.17) is estimated as

$$\begin{aligned}
 -1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} &> 0 \iff B_0 > \frac{4(p-1)}{p} \quad ; \\
 -1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)} &\geq 0 \\
 \iff 0 < \delta \leq \frac{B_0(p-2)}{B_0-2} \left(-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} \right)
 \end{aligned}$$

and then,

$$t_0 - (\Lambda(\rho))^{2-p} \rho^2 \leq \tau \leq t_0 - (\Lambda(r))^{2-p} r^2 \iff r^{B_0} \leq t_0 - \tau \leq \rho^{B_0},$$

$$(t_0 - \tau)^{-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)}} \leq \rho^{B_0 \left(-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)} \right)} \leq 1$$

and thus, the right hand side of (2.17) is bounded above by

$$\frac{1}{B_0} \int_{t_0 - (\Lambda(\rho))^{2-p} \rho^2}^{t_0 - (\Lambda(r))^{2-p} r^2} \|Du(\tau)\|_{L^\infty(\text{supp } \mathcal{B}(\tau))}^{2(p-1)} d\tau.$$

□

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